

SEPARATION IN SIMPLY LINKED NEIGHBOURLY 4-POLYTOPES

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Abstract. The Separation Problem asks for the minimum number $s(O, K)$ of hyperplanes required to strictly separate any interior point O of a convex body K from all faces of K . The Conjecture is $s(O, K) \leq 2^d$ in \mathbb{R}^d , and we verify this for the class of simply linked neighbourly 4-polytopes.

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1 INTRODUCTION

We recall that the Separation Problem is the polar version of the Gohberg-Markus-Hadwiger Covering Problem for convex bodies, and refer to [2], [6] and [9] for an overview of the topic.

For convex d -polytopes P , the Conjecture has been verified in the case that P is cyclic or a type of neighbourly 4-polytope (totally-sewn or with at most ten vertices). We refer to [3] for an overview of these results.

In the following, we assume that P is a neighbourly 4-dimensional polytope in \mathbb{R}^4 . Then P is convex and any two distinct vertices determine an edge of P . We refer to [8] and [12] for the basic geometric and combinatorial properties of P .

With formal definitions to follow; we note only that cyclic polytopes are neighbourly and totally-sewn, and that totally-sewn P are linked. Thus, we verify the Conjecture for a new class of P .

As for organization: Section 2 contains definitions and conventions. In Section 3, we examine the inner structure of P . In Section 4, we determine some separation properties of P . We introduce simply linked P and present our separation results in Section 5 and 6.

2 DEFINITIONS

Let Y be a set of points in \mathbb{R}^d . Then $\text{conv } Y$ and $\text{aff } Y$ denote, respectively, the convex hull and the affine hull of Y . For sets Y_1, Y_2, \dots, Y_k , let

$$[Y_1, Y_2, \dots, Y_k] = \text{conv}(Y_1 \cup Y_2 \cup \dots \cup Y_k)$$

and $\langle Y_1, Y_2, \dots, Y_k \rangle = \text{aff}(Y_1 \cup Y_2 \cup \dots \cup Y_k)$. For a point y , let $[y] = [\{y\}]$ and $\langle y \rangle = \langle \{y\} \rangle$.

Let $Q \in \mathbb{R}^d$ denote a (convex) d -polytope with $\mathcal{V}(Q)$, $\mathcal{E}(Q)$ and $\mathcal{F}(Q)$ denoting, respectively, its sets of vertices, edges and facets. For $x \in \mathcal{V}(Q)$, Q/x denotes the vertex figure of Q at x . For $E = [x, y] \in \mathcal{E}(Q)$, Q/E denotes the quotient polytope $(Q/y)/x$. We note that Q/E is a $(d-2)$ -polytope.

Let $d = 4$. As a simplification, we assume always that Q/x is contained in a hyperplane $H \subset \mathbb{R}^4$ that strictly separates x from each $y \in \mathcal{V}(Q) \setminus \{x\}$, and denote $H \cap [x, y] = H \cap \langle x, y \rangle$ also by y . Then of importance here are the following:

2.1. For $y_i \in \mathcal{V}(Q) \setminus \{x\}$; a plane $\langle y_1, y_2, y_3 \rangle$ separates y_4 and y_5 in $\langle Q/x \rangle$ if, and only if, the hyperplane $\langle x, y_1, y_2, y_3 \rangle$ separates y_4 and y_5 in \mathbb{R}^4 , and

2.2. For $y_i \in \mathcal{V}(Q)/E$; a line $\langle z_1, z_2 \rangle$ separates z_3 and z_4 in $\langle Q/E \rangle$ if, and only if the hyperplane $\langle E, z_1, z_2 \rangle$ separates z_3 and z_4 in \mathbb{R}^4 .

Let $S \subset \mathbb{R}^3$ be a 3-polytope with $s \geq 4$ vertices. Then S is *stacked* if either $s = 4$ or S is the convex hull of a stacked 3-polytope with $s-1$ vertices and a point in \mathbb{R}^3 that is beyond exactly one facet of S .

Let S be stacked, $\{x, y, z\} \subset \mathcal{V}(S)$ and $C = [x, y, z]$ be a triangle. We say that C is a *cut* of S if $\mathcal{E}(C) \subset \mathcal{E}(S)$ but $C \notin \mathcal{F}(S)$. All the cuts of S decompose S into *components*, each of which is a 3-simplex. We note that $|\mathcal{V}(S)| = s$ yields that S has $s-4$ cuts and $s-3$ components.

Let \mathcal{N}_m denote the family of combinatorially distinct neighbourly 4-polytopes with $m \geq 5$ vertices, $P \in \mathcal{N}_{m+1}$, $x \in \mathcal{V}(P)$ and $Q = [\mathcal{V}(P) \setminus \{x\}]$. We note that $Q \in \mathcal{N}_m$.

The relevance of stacked 3-polytopes here is the following result in [1]:

2.3. P/x is a stacked 3-polytope with m vertices; furthermore, $[y_1, y_2, y_3, y_4]$ is a component of P/x if, and only if, $[y_1, y_2, y_3, y_4] \in \mathcal{F}(Q) \setminus \mathcal{F}(P)$. Hence, x is beyond exactly $m-3$ facets of Q .

Next, let $E = [x, y] \in \mathcal{E}(P)$. Then E is a *universal edge* of P if $[E, z]$ is a 2-face of P for each $z \in \mathcal{V}(P) \setminus \{x, y\}$. Let $\mathcal{U}(P)$ denote the *set of universal edges* of P . We observe from [12] and [13] that

2.4. $E = [x, y] \in \mathcal{U}(P)$ if, and only if, x and y lies on the same side of every hyperplane determined by the vertices of P . From the same sources; if $|\mathcal{V}(P)| \geq 7$ then any vertex of P is on at most two members of $\mathcal{U}(P)$, and $|\mathcal{U}(P)| \leq |\mathcal{V}(P)|$.

We recall that a *cyclic 4-polytope* C_m with m vertices is combinatorially equivalent to the convex hull of m points on the moment curve in \mathbb{R}^4 . From [7], [8] and [12], we note that $C_m \in \mathcal{N}_m$, $\mathcal{N}_6 = \{C_6\}$, $|\mathcal{U}(C_6)| = 9$, $\mathcal{N}_7 = \{C_7\}$, $|\mathcal{U}(C_m)| = m$ for $m \geq 7$, and any 4-subpolytope of C_m is again cyclic. For $m \geq 6$, there is a natural ordering (Gale's Evenness Condition) of $\mathcal{V}(P_m)$ that corresponds to the order of appearance of equivalent points on the moment curve.

Let $m \geq 8$. Most of our knowledge about members of \mathcal{N}_m is based upon various *construction techniques*: given $Q \in \mathcal{N}_{m-1}$, find a point $\bar{x} \in \mathbb{R}^4/Q$ such that $\bar{Q} = [Q, \bar{x}] \in \mathcal{N}_m$. It is noteworthy that, at present, known constructions

such as Shemer Sewing, Extended Sewing and Gale Sewing(cf. [12], [10] and [11]) yield that $\mathcal{U}(\bar{Q}) \setminus \mathcal{U}(Q) \neq \emptyset$. We introduce a class of polytopes to reflect this fact.

Let $n \geq 7$ and $P_n \in \mathcal{N}_n$. We say that P_n is *linked* if for $m = n - 1, \dots, 6$, there is a $P_m \in \mathcal{N}_m$ with the property that

$$P_{m+1} \supset P_m \text{ and } \mathcal{U}(P_{m+1}) \setminus \mathcal{U}(P_m) \neq \emptyset$$

We say that P_n is *linked under the (vertex) array* $x_n > x_{n-1} > \dots > x_1$ if for $m = n - 1, \dots, 6$,

$$P_m = [x_m, x_{m-1}, \dots, x_1] \text{ and } \mathcal{U}(P_{m+1}) \setminus \mathcal{U}(P_m) \neq \emptyset$$

For $x_t \in \{x_7, \dots, x_n\}$ and $x_r \in \{x_1, \dots, x_{t-1}\}$, we say that x_t is *linked to* x_r ($x_t \rightarrow x_r$) if $[x_t, x_r] \in \mathcal{U}(P_t)$ and $[x_t, x_j] \notin \mathcal{U}(P_t)$ for $j > \max\{6, r\}$.

By way of clarification for requiring that $t \geq 7$; we note that

2.5. P_6 is cyclic and there are disjoint three element subsets Y and Z of $\mathcal{V}(P_6)$ such that $\mathcal{U}(P_6) = \{[y, z] | y \in Y \text{ and } z \in Z\}$. Thus, there is no meaningful labeling of a greatest or a least vertex of P_6

3 THE INNER STRUCTURE OF P

Let $v \in \mathcal{V}(P)$, $Q \subset P$, $v \notin Q$, $Q \in \mathcal{N}_m$ and $R = [Q, v]$. We recall that $R^* = R/v$ is a stacked 3-polytope and that $y \in \mathcal{V}(Q)$ denotes also $\{y^*\} = \langle v, y \rangle \cap \langle R^* \rangle$. We describe R^* .

Let

$$Y_a = \{y_1, y_2, \dots, y_a\}, \quad z \in \mathcal{V}(Q) \setminus Y_a,$$

$$Z_t = \{z | \langle v, y_1, y_t, y_{t+1} \rangle \text{ strictly separates } y_2 \text{ and } z\}$$

and

$$Z'_t = \{z | \langle v, y_2, y_t, y_{t+1} \rangle \text{ strictly separates } y_1 \text{ and } z\}$$

From 2.1, we have that

- $Z_t \neq \emptyset$ ($Z'_t \neq \emptyset$) if and only if, $[y_1, y_t, y_{t+1}]$ ($[y_2, y_t, y_{t+1}]$) is a cut of R^* .

Hence, we have a generic description of R^* ; cf. the Schlegel diagram in Figure 1.

Next, we observe from 2.3 and 2.1 that

- $\langle y_1, y_2, y_t, y_{t+1} \rangle$ separates v and Q for $t = 3, \dots, a - 1$, and
- $\langle v, y_1, y_2, y_t \rangle$ separates $Z_r \cup Z'_r$ ($r < t$) and $Z_s \cap Z'_s$ ($s \geq t$) for $t = 4, \dots, a - 1$.

From 2.2, we depict these separation properties with respect to $Q/[y_1, y_2]$ and $R/[y_1, y_2]$ in Figure 2.

REMARKS Let $R^* = R/v$ be labeled as above.

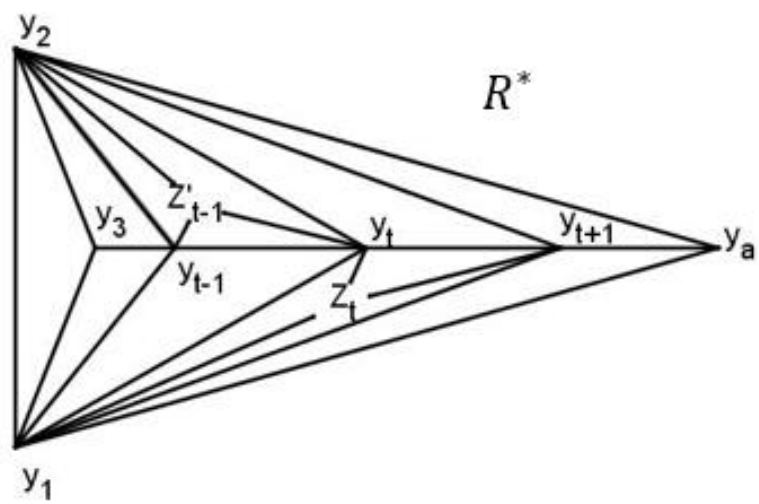


Fig. 1.

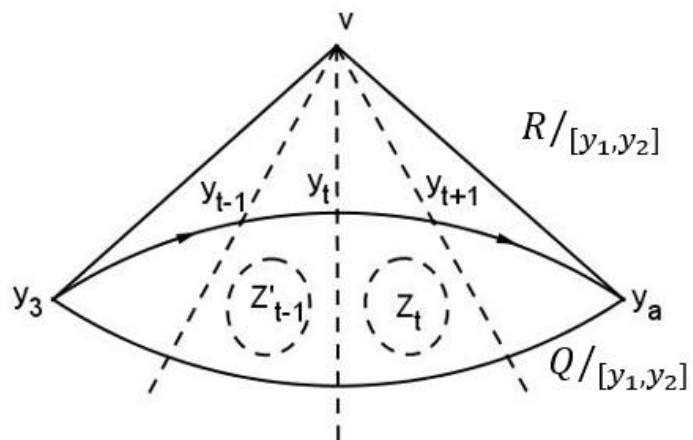


Fig. 2.

3.1. If $a = m$ then $\mathcal{F}(Q) \setminus \mathcal{F}(R) = \{[y_1, y_2, y_t, y_{t+1}] \mid t = 3, \dots, a-1\}$, $[y_1, y_2] \in \mathcal{U}(Q)$ and $\{[v, y_1], [v, y_2]\} \subset \mathcal{U}(R)$.

There is such a labeling of R^* if R is cyclic or if R is constructed by a Shemer sewing of v onto Q .

3.2. If $w \in \mathcal{V}(P) \setminus \mathcal{V}(Q)$ and $[w, v] \in \mathcal{U}([Q, v, w])$ then $\mathcal{F}(Q) \setminus \mathcal{F}([Q, w]) = \mathcal{F}(Q) \setminus \mathcal{F}(R)$ by 2.4.

3.3. Let $3 < t < a$. If $\langle v, y_1, y_2, y_t \rangle$ strictly separates vertices p_t and s_t of Q then $[p_t, s_t]$ is not an edge of R^* , and $[v, p_t, s_t]$ is not a face of R .

We note from Figure 2 that under the hypotheses of 3.3, each hyperplane through $\langle y_1, y_2, y_t \rangle$ strictly separates some two of v, p_t and s_t . Thus, the following is the more general result; cf. [5].

3.4. If $\{x_a, x_b, x_c, x_e, x_f, x_g\}$ is a set of six vertices of P and each hyperplane of \mathbb{R}^4 through $\{x_a, x_b, x_c\}$ strictly separates two of x_e, x_f and x_g , then $[x_a, x_b, x_c]$ and $[x_e, x_f, x_g]$ are not faces of P .

4 GENERIC SEPARATION PROPERTIES OF P

Let $P \in \mathcal{N}_m, m \geq 6$, and O be an interior point of P . We determine hyperplanes $H \in \mathbb{R}^4$ that strictly separate O from facets of P . As a simplification, we determine H that do not contain O . We consider first $F \in \mathcal{F}(P)$ that either are contained in a subpolytope Q such that $O \notin \text{int } Q$ or have a common vertex w .

Lemma A. (cf. [4]) Let $O \in \text{bd}(Q)$. Then O is strictly separated from any $F \in \mathcal{F}(P) \cap \mathcal{F}(Q)$ by one of at most three hyperplanes.

Lemma B. Let $w \in \mathcal{V}(P), R \in \mathcal{N}_{m-1}, P = [R, w]$ and $F \in \mathcal{F}(P)$ such that $w \in F$. Then O is strictly separated from any such F by one of at most four (six) hyperplanes in case O is (is not) an interior point of R .

Proof. Since $P^* = P/w$ is stacked and $O \in \text{int } P$, it follows that $O^* \in \langle w, O \rangle \cap P^*$ is in a component $A^* = [x^*, y^*, z^*, v^*]$ of P^* . If $O^* \in \text{relint } A^*$ then O is separated from F by one of $\langle w, x, y, z \rangle, \langle w, x, y, v \rangle, \langle w, x, z, v \rangle$ and $\langle w, y, z, v \rangle$.

Let $O^* \in B^* = [x^*, y^*, z^*]$, say. Then B^* is a cut of P^* , $O \in \langle w, x, y, z \rangle$ and there are subpolytopes P' and P'' of P such that $P' \cap P'' = [w, x, y, z], [P', P''] = P$ and (since $w \in F$) either $F \subset P'$ or $F \subset P''$.

We recall from 2.3 that $[x, y, z, v] \in \mathcal{F}(R) \setminus \mathcal{F}(P)$. If $O \in \text{int } R$ then it is clear that $O \notin [w, x, y, z]$; that is, $O \notin P' \cup P''$ and O is separated from F by one of two hyperplanes. If $O \in [w, x, y, z]$ then $O \in \text{bd}(P') \cap \text{bd}(P'')$ and we apply LEMMA A. \square

REMARKS Let Q be a subpolytope of P such that $O \notin \text{int } Q$.

4.1. If $Q \in \mathcal{N}_{m-1}$ then O is strictly separated from any $F \in \mathcal{F}(P)$ by one of at most nine (three from LEMMA A, six from LEMMA B) hyperplanes.

4.2. If $Q \in \mathcal{N}_{m-3}$ then O is strictly separated from any $F \in \mathcal{F}(P)$ by one of at most fifteen hyperplanes.

For 4.2, we apply *LEMMA B* under the assumption that O is an interior point of any $Q' \in \mathcal{N}_{m-1}$ such that $Q' \subset P$

5 SIMPLY LINKED P

Let $n \geq 7$ and $P = P_n \in \mathcal{N}_n$ be linked under the array $x_n > x_{n-1} > \cdots > x_1$.

Let $\mathcal{W} = \{w_s, w_{s-1}, \dots, w_1\}$ be an s element subset of $\mathcal{V}(P)$ with the induced array $w_s > w_{s-1} > \cdots > w_1$ in the case $s > 1$. Then \mathcal{W} is a *chain* if either $s = 1$ or

$$w_s \rightarrow w_{s-1} \rightarrow \cdots \rightarrow w_1.$$

For $x_k \in \mathcal{V}(P)$, let \mathcal{V}^k denote the union of all chains of P with x_k as the least vertex.

Finally, we say that P_n is *simply linked* if for $k = 7, \dots, n$:

- \mathcal{V}^k is a chain, and
- for disjoint chains \mathcal{V}^c and \mathcal{V}^d , there are $x_i \neq x_j$ in $\mathcal{V}(P_6)$ such that $x_c \rightarrow x_i, x_d \rightarrow x_j$ and $[x_i, x_j] \notin \mathcal{U}(P_6)$.

Henceforth, we assume that P_n is simply linked. Then it follows from 2.5 that $\{x_7, \dots, x_n\}$ is the union of at most three pairwise disjoint maximal chains.

Lemma C. Let $6 \leq m < n, x_m < x_t, x_k < x_t$ and $x_t \notin \mathcal{V}^k$.

C.1 $H \cap [\mathcal{V}^t] = \emptyset$ for any hyperplane H spanned from $\{x_1, \dots, x_m\}$.

C.2 Let $H_h = \langle x_a, x_b, x_c, x_h \rangle$ be a hyperplane with $\{x_a, x_b, x_c\} \subset \{x_1, \dots, x_m\}$ and $H_h \cap \mathcal{V}^k = \{x_h\}$. Then $H_h \cap [\mathcal{V}^t] = \emptyset$.

C.3 Let $x_t \rightarrow x_j, x_j \notin \{x_a, x_b, x_c\}$ and H_h be defined as above. Then $H_h \cap [\mathcal{V}^j] = \emptyset$.

Proof. Since P is simplicial, it follows from $H \cap [\mathcal{V}^t] \neq \emptyset$ that H strictly separates some x_v and x_u in the chain \mathcal{V}^t such that $x_v \rightarrow x_u$. Then $[x_v, x_u] \in \mathcal{U}(P_v)$ and $P_m \subset P_t \subset P_v$ yield a contradiction by 2.4.

As above, $H_h \cap [\mathcal{V}^t] \neq \emptyset$ implies that H_h strictly separates some x_s and x_q in \mathcal{V}^t such that $x_s \rightarrow x_q$. Thus, C.1 yields that $x_s < x_h$ and $x_t \in P_s \subset P_h$. From $x_h \in \mathcal{V}^k$ and $x_k < x_t < x_h$, there is an $x_g \in \mathcal{V}^k$ such that $x_h \rightarrow x_g$. Then $[x_g, x_h] \in \mathcal{U}(P_h), x_m < x_t < x_h, \mathcal{V}^k \cap \{x_a, x_b, x_c\} = \emptyset$ and 2.4 yield that in the pencil of hyperplanes containing $\langle x_a, x_b, x_c \rangle$:

$$\langle x_a, x_b, x_c, x_s \rangle \cap [x_g, x_h] = \emptyset = \langle x_a, x_b, x_c, x_q \rangle \cap [x_g, x_h].$$

Hence, $\langle x_a, x_b, x_c, x_g \rangle$ also strictly separates x_s and x_q , and $x_t \in P_s \subset P_g \subset P_h$. It now follows from $x_h \rightarrow x_g \rightarrow \cdots \rightarrow x_k$ that $x_t \in P_s \subset P_k \subset P_h$; a contradiction.

We note that $\mathcal{V}^j = \{x_j\} \cup \mathcal{V}^t$ and that if $H_h \cap [\mathcal{V}^j] \neq \emptyset$ then H_h strictly separates x_t and x_j by C.2, and $x_t < x_h$ by C.1. We now argue on above and obtain a contradiction. \square

REMARKS We recall that $P_m = [x_m, x_{m-1}, \dots, x_1]$ for $m = n, \dots, 6$. Let P_5 denote any 4-subpolytope of P_6 . In view of 2.5,

5.1. *there is a labeling of $\mathcal{V}(P_6)$, which we may denote by x_1, x_2, \dots, x_6 , such that*

- P_6 satisfies Gale's Evenness Condition with $x_1 < x_2 < \dots < x_6$, $Y = \{x_1, x_3, x_5\}$, $Z = \{x_2, x_4, x_6\}$
- $P_5 = [x_1, x_2, \dots, x_5]$, and
- any hyperplane through $\langle Y \rangle$ strictly separates two elements of Z .

We recall that $P = P_n$ is simply linked under $x_n > x_{n-1} > \dots > x_1$ and $P_m = [x_m, \dots, x_1]$ for $m \geq 5$. Let O be an interior point of P , $6 \leq m \leq n-1$ and $O \in P_m \setminus P_{m-1}$. We note that a vertex of $[x_{m+1}, \dots, x_n]$ is linked to a vertex of P_m .

With $v = x_w > x_m$, $Q = P_m$ and $R = [Q, v]$, we label Q and $R^* = R/v$ as in Section 3 so that $x_w \rightarrow y_1$ (hence, each Z'_t is empty) and $\langle x_w, O \rangle \cap [y_1, y_2, y_t, y_{t+1}] \neq \emptyset$ for some $3 \leq t \leq a-1$. We let $T = [y_1, y_2, y_t, y_{t+1}]$,

$$\begin{aligned} Z_t^- &= \{y_3, \dots, y_{t-1}\} \cup Z_3 \cup \dots \cup Z_{t-1}, \\ Z_t^+ &= \{y_{t+2}, \dots, y_a\} \cup Z_{t+1} \cup \dots \cup Z_{a-1} \end{aligned}$$

and note that

$$T \in \mathcal{F}(Q) \setminus \mathcal{F}(R) \text{ and } \mathcal{V}(P_m) = Y_a \cup Z_3 \cup \dots \cup Z_{a-1} = \mathcal{V}(T) \cup Z_t^- \cup Z_t \cup Z_t^+.$$

From the Schlegel diagram of R^* on $[y_1, y_2, y_a]$ in Figure 1, we readily obtain diagrams of R^* on 2-faces containing $[y_1, y_t]$ or $[y_2, y_t]$. In Figure 3, 4 and 5, we depict associated polygons $R/[y_1, y_2]$, $R/[y_1, y_t]$ and $R/[y_2, y_t]$ that include $[\mathcal{V}^w]$ (as per 3.2 and C.1) and hyperplanes H_1, H_2, H_3, H_4 and H_5 that separate O and $[\mathcal{V}^w]$. We note that each of H_2, H_3, H_4 and H_5 intersects and supports $[\mathcal{V}^w]$. For $i = 1, \dots, 5$, let H_i^- and H_i^+ denote the open half-spaces of \mathbb{R}^4 determined by H_i with $\mathcal{V}^w \subset H_i \cup H_i^+$.

REMARKS Let $F \in \mathcal{F}(P)$ and assume by 4.2 that $m \leq n-3$. From $\langle x_w, O \rangle \cap T \neq \emptyset$, we have the following:

5.2. *O is separated from all F with a common vertex by one of at most four hyperplanes; cf. LEMMA B.*

5.3. *$O \in [x_m, P_{m-1}] \setminus P_{m-1}$ is separated from any $F \in \mathcal{F}(P_m)$ by one of at most five hyperplanes.*

5.4. *If $F \cap \mathcal{V}^w \neq \emptyset$ then F intersects at most one of Z_t^-, Z_t and Z_t^+ ; cf. 3.4.*

5.5. *If $O \notin T$ then O is separated from any F such that $F \cap \mathcal{V}^w \neq \emptyset$ and $\mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w$ by one of H_2, H_3, H_4 , and H_1 or H_5 in the case $F \cap (Z_t^- \cup Z_t \cup Z_t^+) = \emptyset$.*

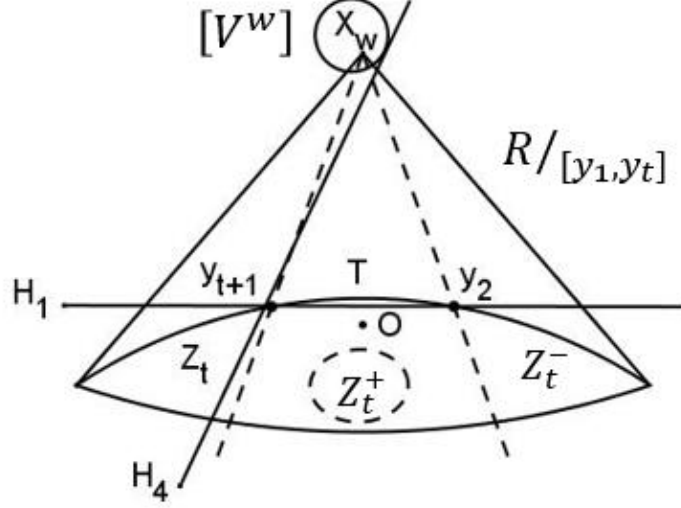


Fig. 4.

- $\langle x_r, O \rangle \cap I \neq \emptyset$ with $I = [v_1, v_2, v_i, v_{i+1}] \notin \mathcal{F}([P_m, x_r])$,
- O is separated from $[\mathcal{V}^s]([\mathcal{V}^r])$ by $\hat{H}_1, \dots, \hat{H}_5(\bar{H}_1, \dots, \bar{H}_5)$ and
- $\mathcal{V}^s \subset \hat{H}_j \cup \hat{H}_j^+ (\mathcal{V}^r \subset \bar{H}_j \cup \bar{H}_j^+)$ for $j = 1, \dots, 5$.

REMARK We refer to Figure 4, and consider any hyperplane H' through $\langle y_1, y_t, y_{t+1} \rangle$ in the case that $x_s \rightarrow u_1 = y_2$. Then $[x_s, y_2] \in \mathcal{U}(P_s)$, and it follows from LEMMA C that if $H' \cap Z_t \neq \emptyset$, then $H' \cap [y_2, x_s] = \emptyset, H' \cap [\mathcal{V}^s] = \emptyset$ and H' strictly separates $[\mathcal{V}^s]$ and $[\mathcal{V}^w]$. The following now follows from 3.4:

5.7. If $u_1 = y_2$ and $F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^w$ then $F \cap Z_t = \emptyset$.

Lemma D. Let $F \in \mathcal{F}(P)$. Then F intersects at most two of $\mathcal{V}^2, \mathcal{V}^4$ and \mathcal{V}^6 , and at most two of $\mathcal{V}^w, \mathcal{V}^s$ and \mathcal{V}^r .

Proof. The existence of $\mathcal{V}^w, \mathcal{V}^s$ and \mathcal{V}^r imply that $\{x_7, \dots, x_n\}$ is the union of pairwise disjoint chains $\mathcal{V}^e, \mathcal{V}^f$ and \mathcal{V}^g , say. Since P_6 is cyclic with $x_1 < x_2 < \dots < x_6$, we may assume by 2.5 and 5.1 that $x_e \rightarrow x_2, x_f \rightarrow x_4$ and $x_g \rightarrow x_6$.

From 5.1 and LEMMA C, we obtain that any hyperplane H through $\langle x_1, x_3, x_5 \rangle$ strictly separates two of $\mathcal{V}^2, \mathcal{V}^4$ and \mathcal{V}^6 . Hence, no face of P intersects each of $\mathcal{V}^2, \mathcal{V}^4$ and \mathcal{V}^6 by 3.4. \square

REMARK We refer to Figure 3, 4 and 5, and consider a $v \in \mathcal{V}(P)$ with the property that $v \in H_1^-, v \notin H_2^+ \cup H_3^+ \cup H_4^+$ and no hyperplane spanned from $v, y_1, y_2, y_t, y_{t+1}$ intersects $[\mathcal{V}^w]$.

Then $\langle x_w, y_1, y_2, y_t \rangle$ separates v and $Z_t^-, \langle x_w, y_1, y_2, y_{t+1} \rangle$ separates v and Z_t^+ , and $\langle x_w, y_1, y_t, y_{t+1} \rangle$ separates v and Z_t . From $[P_m, x_w]/[x_w, y_1]$, it now follows that $[x_w, y_1, v]$ is not a 2-face of $[P_m, x_w, v]$ or $[x_w, y_1, y_2, y_t, y_{t+1}, v]$. Since the latter polytope is cyclic, we obtain from 5.1 that

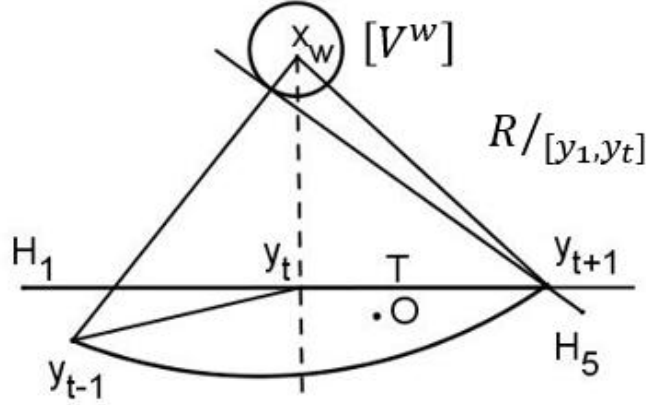


Fig. 5.

5.8. $\langle v, y_2, y_t, y_{t+1} \rangle$ strictly separates x_w and y_1 .

Lemma E. Let $x_w < x_s, F \in \mathcal{F}(P)$ and $F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s$. Then O is separated from any such F by

E.1 at most three hyperplanes ($\hat{H}_2, \hat{H}_3, \hat{H}_4$) in the case $x_w \in \hat{H}_1^-$ and $O \notin bd(K)$,

E.2 one hyperplane ($H_i, 2 \leq i \leq 4$) in the case $u_1 \notin T$ and $O \notin bd(T)$,

E.3 one hyperplane ($H_i, 2 \leq i \leq 5$) in the case $x_w \in \hat{H}_i^+, u_1 \in T, x_s \in H_1^-$ and $O \notin bd(T)$, and

E.4 at most two hyperplanes from $H_2, H_3, \hat{H}_2, \hat{H}_3$ in the case $x_w \in \hat{H}_1^+, x_s \in H_1^+, O \notin bd(K) \cup bd(T)$ and either $T \neq K$, or $T = K$ and $F \cap H_1^- \neq \emptyset \neq F \cap \hat{H}_2^-$.

Proof. We refer to Figure 3, 4 and 5, and the analogous figures with $\mathcal{V}^s, K, U_j = \hat{Z}_j$ and \hat{H}_j for the location of O , and note $\mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w \cup \mathcal{V}^f$ by LEMMA D.

E.1 Let $x_w \in \hat{H}_1^-$. Then $\mathcal{V}^w \subset \hat{H}_1^-$ by C.1, and either $(\mathcal{V}(F) \cap \hat{H}_1^-) \cap \hat{H}_j^+ \neq \emptyset$ for some $j \in \{2, 3, 4\}$ or there is a $v \in \mathcal{V}^w$ such that $v \notin \hat{H}_2^+ \cup \hat{H}_3^+ \cup \hat{H}_4^+$. In case of the former, O is separated from F by \hat{H}_j ; cf. 5.4. In case of the latter, it follows from $x_w < x_s$ and C.3 that $\langle v, u_2, u_k, u_{k+1} \rangle \cap [\mathcal{V}^s \cup \{u_1\}] = \emptyset$; a contradiction of 5.8.

E.2 Let $u_1 \notin T$. Then $x_s \rightarrow u_1 \in P_m$ yields that $u_1 \in Z_t^- \cup Z_t \cup Z_t^+ \subset H_2^+ \cup H_3^+ \cup H_4^+$ and $\mathcal{V}^s \subset H_1^-$. Now $x_w < x_s$ and C.1 yield that if $u_1 \in H_j^+$ then $\mathcal{V}^s \subset H_j^+$ and O is separated from F by H_j .

E.3 Let $x_w \in \hat{H}_1^+$ and $u_1 \in T$. Then $u_1 \in \{y_2, y_y, y_{t+1}\}, y_1 \in \{u_2, u_k, u_{k+1}\}$ and may assume that $u_1 = y_2$ and $y_1 = u_2$. From 5.7, we obtain that $F \cap Z_t = \emptyset = F \cap U_k$.

Let $x_s \in H_1^-$. Then $\langle T \rangle = H_1 \neq \hat{H}_1 = \langle K \rangle$ and $T \neq K$; cf. Figure 6 with $x_s \in H_3^+$, say, and $\mathcal{V}^s \subset H_1^- \cap H_3^+$. We consider the hyperplanes through

E.4 Let $x_w \in \hat{H}_1^+$, $x_s \in H_1^+$ and $O \notin bd(K) \cup bd(T)$. Then $u_1 \in T$, we assume that $(y_1, y_2) = (u_2, u_1)$ and note that $F \cap (Z_t \cup U_k) = \emptyset$.

Let $Z_t^-(s) = \{z \in Z_t^- \mid \langle y_1, y_2, y_{t+1}, z \rangle \text{ does not separate } x_w \text{ and } x_s\}$. We apply C.1 and 3.4, and obtain that $F' \cap Z_t^- \subset Z_t^-(s)$. Thus, either O is separated from F' by H_1 (and so \hat{H}_2), or there is an F' such that $F' \cap Z_t^- \neq \emptyset$. In the latter case, it is easy to check (cf. Figure 6) that O is separated from any such F' by H_2 or \hat{H}_2 .

REMARK We observe that under the hypotheses of LEMMA E, it follows from LEMMA A that

Fig. 6.

6 SEPARATION RESULTS

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assumption that $P = P_n$ is simply linked under the array $x_n > x_{n-1} > \cdots > x_1$, $P_m = [x_m, \cdots, x_2, x_1]$ for $m = n-1, \cdots, 5$ and $\mathcal{V}(P) = \{x_1, x_3, x_5\} \cup \mathcal{V}^2 \cup \mathcal{V}^4 \cup \mathcal{V}^6$.

6.1. *We consider first the case of $O \in P_m \setminus P_{m-1}$ for some $6 \leq m \leq n$. As noted in Sections 4 and 5, we may assume that $m \leq n-3$ and that $\{x_{m+1}, \cdots, x_n\}$ is the union of non-empty chains $\mathcal{V}^r, \mathcal{V}^s$ and \mathcal{V}^w described in Section 5.*

Our arguments are based upon

- the location of x_m with respect to T and K ,
- the order of x_r with respect to $x_w < x_s$, and
- the location of O with respect to T, K and I .

For each location of O , we present the separation result

- $\{k\}$: property: rationale

to indicate that at most k separating hyperplanes suffice for $F \in \mathcal{F}(P)$ with the indicated property due to the specified reasons. $\{-\}$ indicates that the separating hyperplanes for this case have already counted.

I. $x_m \notin T \cup K$

Then $T \cup K \subset P_{m-1}$ and $O \notin T \cup K$. From $x_m \in H_1^- \cap \hat{H}_1^-, x_r \rightarrow x_m$ and 2.4, we have that $x_r \in H_1^- \cap \hat{H}_1^-$. Next, LEMMA D and its proof yield that any F intersects at most two of $\mathcal{V}^w, \mathcal{V}^r$ and \mathcal{V}^s , and that $[v_1, u_1, y_1]$ is not a 2-face of P_m . Hence, $x_m \in I$ implies that $\{u_1, y_1\} \not\subset I$.

I.1 $O \notin bd(I)$

We apply our Lemmas and Remarks. Then

- $\{4\} : x_m \in F : 5.2$
- $\{1\} : \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^w : 5.6$
- $\{1\} : \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^s : 5.6$
- $\{3\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : E.1$, and
- $\{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 5.5$ with $\bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5$ (as $v_1 = x_m \notin F$).

It remains to consider F that intersect \mathcal{V}^r and $\mathcal{V}^w \cup \mathcal{V}^s$. Here, we apply $\{u_1, y_1\} \not\subset I$ and LEMMA E with relabeling as necessary

I. 1.1 $x_r < x_w < x_s$

As x_w and x_s are interchangeable with respect to x_r , we assume that $u_1 \notin I$, say. Then

- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r < x_s, u_1 \notin I$ and E.2 with \bar{H}_2, \bar{H}_3 and \bar{H}_4 already counted, and

- $\{3\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : x_r < x_w, x_r \in H_1^-$ and E.1.

I. 1.2 $x_w < x_r < x_s$

If $u_1 \notin I$ then one case is above, and

- $\{1\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : x_w < x_r, v_1 = x_m \notin T$ and E.2

If $u_1 \in I$ and $y_1 \notin I$ then

- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : x_w < x_r, x_w \in \bar{H}_1^-$ and E.1, and
- $\{3\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r < x_s, x_r \in \hat{H}_1^-$ and E.1 .

I. 1.3 $x_w < x_s < x_r$

Then $v_1 = x_m \notin T \cup K$ and E.2 yield $\{1\}$ for $F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r$, and $\{1\}$ for $F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r$.

I.2 $O \in bd(I)$

We recall that $\langle x_w, O \rangle \cap T \neq \emptyset$ and $O \notin T$. Hence, $H_1 = \langle T \rangle$ strictly separates O and x_w , and x_w is necessarily beneath any facet of P_m that contains O . Thus $x_w \in \bar{H}_1^-$ and, similarly, $x_s \in \bar{H}_1^-$; whence $\mathcal{V}^w \cup \mathcal{V}^s \subset \bar{H}_1^-$. Since $v_1 = x_m$ implies that $\bar{H}_1 \cap \bar{H}_5 = [v_2, v_i, v_{i+1}] \subset bd(P_{m-1})$, it follows from $O \in \bar{H}_1 \setminus P_{m-1}$ that $O \notin [v_2, v_i, v_{i+1}]$. From these observations, we have that

- $\{3\} : F \cap \mathcal{V}^r = \emptyset : (LEMMA)A$.
- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset$: 5.5, A and 5.9 with $\bar{H}_1, \bar{H}_2, \bar{H}_3, \bar{H}_4$ as separating hyperplanes and $O \in \bar{H}_1 \cap \bar{H}_j$ for some $j \in \{2, 3, 4\}$. We apply LEMMA A and replace \bar{H}_1 and \bar{H}_j as per 5.9. We indicate these eight hyperplanes by $2\bar{H}_i + 3 + 3$.

We now argue as in I.1.1, I.1.2 and I.3 with 5.9 applied for \bar{H}_1 and \bar{H}_j , and obtain the same counts. Thus, $s(O) \leq 16$ in each of these cases.

II. $x_m \in K$ and $x_m \notin T$.

Then $T \subset P_{m-1}, O \notin T, x_r \in H_1^-$ and we let $x_m = u_2$. We have again that $\{y_1, u_1\} \not\subset I$; and from $\{v_1, u_1\} \subset K$, it follows that $y_1 \notin K$ and $x_w \in \hat{H}_1^-$. We note that $x_m = u_2$ yields that \hat{H}_4 separates O from any F with $x_m \notin F$ and $\mathcal{V}(F) \subset \mathcal{V}^s \cup \{u_1, u_2, u_k, u_{k+1}\}$.

II.1 $O \notin bd(I) \cup bd(K)$

Similarly to I.1, we obtain

- $\{4\} : x_m \in F$: 5.2,
- $\{1\} : \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^s$: 5.6,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cap \mathcal{V}^s \neq \emptyset$: 5.5 with $\hat{H}_2, \hat{H}_3, \hat{H}_4$,

- $\{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 5.5$ with $\bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5$ and
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \hat{H}_1^-,$ and E.1.

II.1.1 $x_r < x_w < x_s$

If $u_1 \notin I$ then we recall that $x_r \in H_1^-$ and argue as in I.1. If $y_1 \notin I$ then

- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : x_r < x_w$ and E.2.

For $F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$; we obtain from $x_r < x_s$ and $u_1 \in I$ that one of E.1, E.3 or E.4 is applicable. We note that E.1 and E.3 yield $\{-\}$, and E.4 yields either $\{-\}$ or $\{3\}$ with $O \in I = K$ and 5.9 applied to $\hat{H}_1 = \bar{H}_1$.

Henceforth, as a simplification, we list only “worst case scenario” results. In that regard, it is noteworthy that the assertion of E.4 is the same if x_s and x_w are interchanged.

II.1.2 $x_w < x_r < x_s$ or $x_w < x_s < x_r$

Then as worst case scenarios, we have

- $\{1\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : x_m = v_1 \notin T$ and E.2, and
- $\{3\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r : \text{E.4, 5.9 with } O \in \hat{H}_1 = \bar{H}_1.$

II.2 $O \in bd(I)$

We note that as in I.2; $x_w \in \bar{H}_1^-$ and

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cap \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 2\bar{H}_i + 3 + 3$ with $O \in \bar{H}_1 \cap \bar{H}_j$ for some $j \in \{2, 3, 4\}$.

If $x_s \in \bar{H}_1^-$ then $\mathcal{V}^w \cup \mathcal{V}^s \subset \bar{H}_1^-$,

- $\{3\} : F \cap \mathcal{V}^r = \emptyset$: LEMMA A

and, as worst case scenario, E.2 and $\{y_1, u_1\} \not\subset I$ yield $x_r < x_w < x_s$ and $u_1 \in I$. Then

- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : y_1 \notin I$ and E.2, and
- $\{5\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r \in \hat{H}_1^-, \text{ E.1, 5.9 with } O \in \hat{H}_i \text{ for some } i \in \{2, 3, 4\}$

Let $x_s \in \bar{H}_1^+$. Then $u_1 \in I$ with $u_1 = v_2$, say, and $\langle x_s, O \rangle \cap bd(P_m) \subset I$. Hence, we choose $K = I$ with $u_2 = v_1, u_k = v_i$ and $u_{k+1} = v_{i+1}$. Then $\hat{H}_1 = \bar{H}_1$ and

- $\{5\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cup \mathcal{V}^s \neq \emptyset : 2\hat{H}_i + 3$ with $O \in \bar{H}_1 \cap \hat{H}_j$ and $\{\hat{H}_j, \hat{H}_i, \hat{H}_i\} = \{\hat{H}_2, \hat{H}_3, \hat{H}_4\}$.

From $\{y_1, u_1\} \not\subset I$, we obtain that $y_1 \notin I$ and $x_w \in \bar{H}_1^- = \hat{H}_1^-$ and

- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w$: LEMMA A,

- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \hat{H}_1^-$ and E.1,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : \text{either } x_w < x_r \text{ and E.1, or } x_r < x_w, y_1 \notin I \text{ and E.2, and}$
- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : \text{E.4 with } \bar{H}_1 = \hat{H}_1, \bar{H}_2, \bar{H}_3, \hat{H}_2, \hat{H}_3.$

II.3 $O \in bd(K)$ and $O \notin bd(I)$

We recall that $x_w \in \hat{H}_1^-$, and note that $I \neq K$ implies that $x_r \in \hat{H}_1^-$. Then

- $\{3\} : F \cap \mathcal{V}^s = \emptyset$: LEMMA A,
- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cap \mathcal{V}^s \neq \emptyset : 2\hat{H}_i + 3 + 3$ with $O \in \hat{H}_1 \cap \hat{H}_j$ for some $j = \{2, 3, 4\}$,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \hat{H}_1^-$ and E.1, and as worst case scenario,
- $\{3\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r : x_s < x_r, x_s \in \bar{H}_1^-$ and E.1.

III. $x_m \in T$ and $x_m \notin K$.

As $\{y_1, v_1 = x_m\} \subset T$ and $K \subset P_{m-1}$, we have that $u_1 \in H_1^-, \mathcal{V}^s \subset H_1^-, \mathcal{V}^r \subset \hat{H}_1^-$ and $O \notin K$. We let $v_1 = y_2$ and note that H_4 separates O from any F with $x_m \notin F \subset [\mathcal{V}^w \cup \{y_1, y_t, y_{t+1}\}]$.

III.1. $O \notin bd(I) \cup bd(T)$

As in II.1, we obtain that

- $\{4\} : x_m \in F$: 5.2,
- $\{1\} : \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^s$: 5.6,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset$: 5.5 with H_2, H_3, H_4 ,
- $\{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset$: 5.5 with $\bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5$, and
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, u_1 \neq T$ and E.2.

We note that our repetitive arguments are dependent upon Lemmas A and E, and $\{u_1, y_1\} \not\subset I$. Also that we present only worst case scenarios.

If $x_r < x_s$ then with $u_1 \in I$ and $y_1 \notin I$, we have

- $\{3\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r \in \hat{H}_1^-$ with E.1, and
- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : \text{either } x_r < x_w \text{ with E.2, or } x_w < x_r \text{ with E.1.}$

Let $x_w < x_s < x_r$. By E.1, we may assume that $\{x_w, x_s\} \not\subset \bar{H}_1^-$. With $x_s \in \bar{H}_1^+$ and $x_r \in \hat{H}_1$, we have $u_1 \in I, y_1 \notin I, x_w \in \bar{H}_1^-$,

- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r$: E.1, and
 - $\{1\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r$: E.2 or E.3.
- With $x_w \in \bar{H}_1^+$, we have $x_s \in \bar{H}_1^-$,
- $\{-\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r$: E.1, and
 - $\{3\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r$: E.4 and 5.9 with $O \in T = I$.

III.2 $O \in bd(I)$.

We note as in I.2 that $x_s \in \bar{H}_1^-$ follows from $O \notin K$. Next, we obtain the same separating hyperplanes for F with $F \cap \mathcal{V}^r \neq \emptyset$ and $\mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r$ as in II.2, and with x_w and x_s interchanged, the corresponding worst case scenario for $x_w \in \bar{H}_1^-$.

Let $x_w \in \bar{H}_1^+$. Then we choose $T = I$ and, similarly to II.2, obtain that

- $\{5\} : \mathcal{V}(P) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset : 2H_i + 3$ with $H_1 = \bar{H}_1$, and $O \in H_1 \cap H_j$ for some $j \in \{2, 3, 4\}$,
- $\{3\} : \mathcal{V}(P) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s$: LEMMA A,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, u_1 \notin T$ and E.2,
- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: either $x_r < x_s, u_1 \notin I$ and E.2, or $x_s < x_r, x_s \in \bar{H}_1^-$ and E.1, and
- $\{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: E.4.

III.3 $O \in bd(T)$ and $O \notin bd(I)$.

Then $I \neq T$ and $x_r \in H_1^-$. We recall that $y_1 \notin T$ and $x_s \in H_1^-$. Hence,

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset : 2H_i + 3 + 3$ with $O \in H_1 \cap H_j$ and $\{H_j, H_i, H_i\} = \{H_2, H_3, H_4\}$.
- $\{3\} : F \cap \mathcal{V}^w = \emptyset$: LEMMA A,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, u_1 \notin T$ and E.2, and as worst case scenario,
- $\{3\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : x_w < x_r, x_w \in \bar{H}_1^-$ and E.1.

IV $x_m \in T \cap K$.

We let $v_1 = x_m = y_2 = u_2$, and note that $\{v_1, y_1\} \subset T$ implies that $u_1 \notin T$ and $x_s \in H_1^-$; and $\{v_1, u_1\} \subset K$ implies that $y_1 \notin K$ and $x_w \in \bar{H}_1^-$.

IV.1 $O \notin bd(K) \cup bd(T) \cup bd(I)$.

We recall that $O \in P_m \setminus P_{m-1}$. Then

- $\{5\} : x_m \in F$ or $F \subset P_m$: 5.3,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w \neq \emptyset, y_2 = x_m \notin F$: 5.5 with H_2, H_3 , and H_4 for $F \cap Z_t \neq \emptyset$ or $F \cap (Z_t^- \cup Z_t \cup Z_t^+) = \emptyset$,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cap \mathcal{V}^s \neq \emptyset, u_2 = x_m \notin F$: 5.5 with $\hat{H}_2, \hat{H}_3, \hat{H}_4$,
- $\{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset$: 5.5 with $\bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5$, and
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \hat{H}_1^-$ and E.1.

We observe that for x_r and x_s : E.1 and E.2 yield $\{-\}$ for $F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$, and E.3 and E.4 yield $u_1 \in I, v_1 \in K$ and the worst case scenario

- $\{1\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: either $x_s < x_r$ with \hat{H}_5 , or $x_r < x_s$ with $\bar{H}_1 = \hat{H}_1$.

The corresponding observation for x_r and x_w , and $\{u_1, y_1\} \not\subset I$, now yield

- $\{1\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap (\mathcal{V}^w \cup \mathcal{V}^s)$.

IV.2 $O \in bd(K)$.

Then $O \notin P_{m-1}$ and $u_2 = x_m$ imply that $O \notin [u_1, u_k, u_{k+1}]$ and

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cap \mathcal{V}^s \neq \emptyset : 2\hat{H}_i + 3 + 3$ with $O \in \hat{H}_1 \cap \hat{H}_j$ and $\{\hat{H}_j, \hat{H}_i, \hat{H}_i\} = \{\hat{H}_2, \hat{H}_3, \hat{H}_4\}$

We recall that $x_w \in \hat{H}_1^-$. If $x_r \in \hat{H}_1^-$ then

- $\{3\} : F \cap \mathcal{V}^s = \emptyset$: LEMMA A,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s$ and E.1, and
- $\{3\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_s < x_r$ and either $x_s \in \bar{H}_1^-$ and E.1, or $x_s \in \bar{H}_1^+, v_1 \in K, x_r \in \hat{H}_1^-$ and E.3, 5.9 with $O \in \hat{H}_5$.

Let $x_r \in \hat{H}_1^+$. Then we choose $I = K$ with $(v_1, v_2, v_i, v_{i+1}) = (u_2, u_1, u_k, u_{k+1})$, and note that $y_1 \in I = K$ and $x_w \in \bar{H}_1^- = \hat{H}_1^-$. Now

- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w$: LEMMA A with \hat{H}_1^- ,
- $\{5\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 2\bar{H}_i + 3$ with $O \in \hat{H}_1 \cap \bar{H}_j$ and $\{\bar{H}_j, \bar{H}_i, \bar{H}_i\} = \{\bar{H}_2, \bar{H}_3, \bar{H}_4\}$,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s$ and E.1,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r$: either $x_w < x_r$ and E.1, or $x_r < x_w$ and E.2, and
- $\{-\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r$: E.4.

IV.3 $O \in bd(T)$

We argue as in IV.2 with

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset : 2H_i + 3 + 3$,
- $\{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s$ and E.2, and the cases $x_r \in H_1^-$ and $x_r \in H_1^+$.

IV.4 $O \in bd(I)$ and $O \notin bd(K) \cup bd(T)$

Since we choose $K = I(T = I)$ if $x_s \in \bar{H}_1^+(x_w \in \bar{H}_1^+)$, we may assume that $\{x_w, x_s\} \subset \bar{H}_1^-$. Then

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 2\bar{H}_1 + 3 + 3$ with $O \in \bar{H}_1 \cap \bar{H}_j$ and $\{\bar{H}_j, \bar{H}_i, \bar{H}_i\} = \{\bar{H}_2, \bar{H}_3, \bar{H}_4\}$, and
 - $\{3\} : F \cap \mathcal{V}^r = \emptyset$: LEMMA A
- If $u_1 \notin I$, then
- $\{-\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r$: either $x_r < x_s$ and E.2, or $x_s < x_r$ and E.1, and
 - $\{3\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : x_r < x_w$ and E.1.

Let $u_1 \in I$. Then $y_1 \notin I$ and we argue as above with x_s and x_w interchanged. \square

6.2. *It remains to determine $s(O)$ in the case that $O \in P_6$ and (in view of 5.1) O is contained in every 4-subpolytope of P_6 . Since P_6 satisfies Gale's Evenness Condition with $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ and O is not contained in any facet of P_6 , it follows that*

$$O \in [x_1, x_3, x_5] \cap [x_2, x_4, x_6].$$

From LEMMA D and its proof, we have

- $\mathcal{V}(P) = \{x_1, x_3, x_5\} \cup \mathcal{V}^2 \cup \mathcal{V}^4 \cup \mathcal{V}^6$,
- any hyperplane through $\langle x_1, x_3, x_5 \rangle$ intersects at most two of $\mathcal{V}^2, \mathcal{V}^4$ and \mathcal{V}^6 , and
- any $F \in \mathcal{F}(P)$ intersects at most two of $\mathcal{V}^2, \mathcal{V}^4$ and \mathcal{V}^6 .

Let $\mathcal{W}^{ij} = [\{x_1, x_3, x_5\} \cup \mathcal{V}^i \cup \mathcal{V}^j], i \neq j \text{ in } \{2, 4, 6\}$. Then $[x_1, x_3, x_5] \subset bd(\mathcal{W}^{ij})$, any $F \in \mathcal{F}(P)$ is contained in some \mathcal{W}^{ij} , and $s(O) \leq 9$ by LEMMA A. \square

We conclude with the observation that any linked P with $|\mathcal{V}(P)| \leq 11$ is simply linked, and the problem: Is every linked P also simply linked?

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